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Continuity of a Quantum Stochastic Process

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Abstract We investigate a quantum counterpart of the classical notion of a stochastic process continuous with probability one, and prove that the L^2-limit of quantum martingales 'continuous with probability one' is a quantum martingale 'continuous with probability one'. Applications of this result to a number of concrete situations is presented.

Keywords Quantum stochastic process · Segal's continuity · Martingales · von Neumann algebras

1 Introduction

The importance of the notion of continuity with probability one of a stochastic process, in the classical theory of stochastic processes, is widely appreciated. However, due to the obvious reference of this notion to the underlying probability space, it is by no means clear how to generalize it to the case of quantum stochastic processes which are defined as collections of 'quantum random variables', i.e. collections of operators belonging to a von Neumann algebra or, more generally, to some L^p-space over this algebra, where nothing like a probability space turns up. Nevertheless, a deeper analysis of this notion in the spirit of Egorov's theorem reveals its algebraic nature, making it amenable to a natural generalization to the quantum context. Moreover, it turns out that many properties of this 'quantum continuity with probability one' are inherited from its classical ancestor, for instance, the celebrated Kolmogorov theorem on continuity of a stochastic process has its quantum counterpart. In the paper we show that another classical result holds true in the quantum case. Namely, it is proved that the limit in the space L^2 over a von Neumann algebra, of a sequence of martingales 'continuous with probability one' is a martingale 'continuous with probability one'. This result is then applied to a number of situations giving, in particular, 'continuity with probability one' of quantum stochastic integrals with the integrators being continuous martingales, in full accordance with the classical case.

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2 Preliminaries and Notation

A *noncommutative stochastic base* which is a basic object of our considerations consists of the following elements: a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, a normal faithful unital trace τ on $\mathcal{A}$, and a filtration $(\mathcal{A}_t\colon\ t \in [0, +\infty))$, which is an increasing ($s \le t$ implies $\mathcal{A}_s \subset \mathcal{A}_t$) family of von Neumann subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_\infty = (\bigcup_{t\ge 0} \mathcal{A}_t)''$. Then for each t, there exists a normal conditional expectation $\mathbb{E}_t$ from $\mathcal{A}$ onto $\mathcal{A}_t$ such that

$$\tau \circ \mathbb{E}_t = \tau. \tag{1}$$

Moreover, for any $x \in \mathcal{A},\ t \in [0, +\infty)$, we have

$$\mathbb{E}_t |x|^2 \ge |\mathbb{E}_t x|^2. \tag{2}$$

For $t \in [0, +\infty],\ L^2(\mathcal{A}_t)$ will be the non-commutative Lebesgue space associated with $\mathcal{A}_t$ and τ. The theory of such noncommutative L^p-spaces is described e.g. in [13]; for our purposes we only need the case $p = 2$. Recall that $L^2(\mathcal{A}_t)$ consists of densely defined operators on $\mathcal{H}$, affiliated with $\mathcal{A}_t$, and that $L^2(\mathcal{A}_t)$ is completion of $\mathcal{A}_t$ with respect to the norm

$$\|X\|_2 = [\tau(|X|^2)]^{1/2};$$

moreover, for $a \in \mathcal{A}_t$, $X \in L^2(\mathcal{A}_t)$ the operators aX and Xa belong to $L^2(\mathcal{A}_t)$. The conditional expectations $\mathbb{E}_t$ extend to projections of norm one from $L^2(\mathcal{A})$ onto $L^2(\mathcal{A}_t)$ enjoying properties (1) and (2). We shall use the same symbol for these extended conditional expectations.

An $\mathcal{A}$-(respectively L^2-) valued *quantum stochastic process* is a map X from $[0, +\infty)$ into $\mathcal{A}$ (respectively $L^2(\mathcal{A})$). By analogy with the classical case we shall write $X = (X(t)\colon\ t \in [0, +\infty))$ (a family of 'quantum random variables').

An L^2-process $(X(t)\colon\ t \in [0, +\infty))$ is called a *martingale* if for each $s, t \in [0, \infty)$, $s \le t$, we have $\mathbb{E}_s X(t) = X(s)$.

3 Continuity of a Quantum Stochastic Process

The analysis of the notion of continuity with probability one of a classical stochastic process presented below, has already been performed in [10]. To make the paper self-contained we recall it here.

Let $(X(t, \cdot)\colon\ t \in [a, b])$ be a stochastic process over a probability space $(\Omega, \mathcal{F}, P)$. Consider the following condition: for each $\varepsilon > 0$ there is $\Omega_\varepsilon \in \mathcal{F}$ with $P(\Omega_\varepsilon) > 1 - \varepsilon$, such that the paths $\{X(\cdot, \omega)\colon\ \omega \in \Omega_\varepsilon\}$ are equally uniformly continuous. This can be rewritten as:

for each $\varepsilon > 0$ there is $\Omega_\varepsilon \in \mathcal{F}$ with $P(\Omega_\varepsilon) > 1 - \varepsilon$,
having the property:
for each $\eta > 0$ there is $\delta > 0$ such that for any $\omega \in \Omega_\varepsilon$ (*)
and any $s, t \in [a, b]$ with $|t - s| < \delta$,
we have $|X(t, \omega) - X(s, \omega)| \le \eta$.

If the above condition is satisfied, then the paths of the process are uniformly continuous with probability one. Indeed, take $\varepsilon = 1/n$, and let $\Omega_\varepsilon = \Omega_{1/n}$ be as above. Put

$$\Omega_0 = \bigcup_{n=1}^{\infty} \Omega_{1/n}.$$

Then $P(\Omega_0) = 1$, and for each $\omega \in \Omega_0$ we have $\omega \in \Omega_{1/n}$ for some n, which means that the path $X(\cdot, \omega)$ is uniformly continuous.

Now let us assume that the paths are uniformly continuous with probability one, and let $\Omega_0 = \{\omega\colon X(\cdot,\omega)$ is uniformly continuous$\}$. We have $P(\Omega_0) = 1$, and

$$\begin{aligned}\Omega_0 &= \bigcap_{r=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \left\{\omega\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\} \\ &= \bigcap_{r=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \left\{\omega \in \Omega_0\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\}.\end{aligned}$$

The continuity of the paths for $\omega \in \Omega_0$ implies that

$$\begin{aligned}&\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \left\{\omega \in \Omega_0\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\} \\ &\quad= \bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]\cap\mathbb{Q}}} \left\{\omega \in \Omega_0\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\},\end{aligned}$$

where $\mathbb{Q}$ stands for the rational numbers. It follows that the set

$$\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \left\{\omega \in \Omega_0\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\}$$

is measurable, and for each positive integer r we have

$$\begin{aligned}1 &= P\left(\bigcup_{m=1}^{\infty} \bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \left\{\omega\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\}\right) \\ &= \lim_{m\to\infty} P\left(\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \left\{\omega \in \Omega_0\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\}\right).\end{aligned}$$

For any $\varepsilon > 0$ and positive integer r choose m_r such that

$$P\left(\bigcap_{\substack{|t-s|<1/m_r\\ s,t\in[a,b]}} \left\{\omega \in \Omega_0\colon |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\}\right) > 1 - \frac{\varepsilon}{2^r},$$

and put

$$\Omega_\varepsilon = \bigcap_{r=1}^{\infty} \bigcap_{\substack{|t-s|<1/m_r \\ s,t\in[a,b]}} \left\{\omega \in \Omega_0\colon\ |X(t,\omega) - X(s,\omega)| \le \frac{1}{r}\right\}.$$

Then $P(\Omega_\varepsilon) > 1-\varepsilon$. For arbitrary fixed $\eta > 0$ let r_0 be such that $1/r_0 \le \eta$. Put $\delta = 1/m_{r_0}$. For each $\omega_0 \in \Omega_\varepsilon$ we have, in particular, that $\omega_0 \in \{\omega \in \Omega_0\colon\ |X(t,\omega) - X(s,\omega)| \le \frac{1}{r_0}\}$ for any $s,t \in [a,b]$ with $|t-s| < 1/m_{r_0} = \delta$, which means that

$$|X(t,\omega_0) - X(s,\omega_0)| \le \frac{1}{r_0} \le \eta,$$

showing that condition (*) holds.

We have thus shown the equivalence of uniform continuity of paths of the process with probability one and condition (*). Since in our case the uniform continuity of paths is equivalent to ordinary continuity, condition (*) can be treated simply as another definition of the classical notion of a continuous stochastic process.

Let us observe that condition (*) can be given the following form. Denote by χ_E the indicator function of the set E. Then (*) becomes:

for each $\varepsilon > 0$ there is $\Omega_\varepsilon \in \mathcal{F}$ with $P(\Omega_\varepsilon) > 1-\varepsilon$, having the property: for each $\eta > 0$ there is $\delta > 0$ such that for any $s,t \in [a,b]$ with $|t-s| < \delta$, we have

$$\begin{aligned}\sup_{\omega\in\Omega_\varepsilon} |X(t,\omega) - X(s,\omega)| &= \sup_{\omega\in\Omega}[|X(t,\omega) - X(s,\omega)|\chi_{\Omega_\varepsilon}(\omega)] \\ &= \|[X(t,\cdot) - X(s,\cdot)]\chi_{\Omega_\varepsilon}\|_\infty \le \eta.\end{aligned}$$

The above form is essentially *algebraic*, referring only to the algebra $L^\infty(\Omega)$, which becomes clear if we replace the inequality $P(\Omega_\varepsilon) > 1-\varepsilon$ by the equivalent inequality $\int_\Omega \chi_{\Omega_\varepsilon} dP > 1-\varepsilon$. Thus for a quantum process $(X(t)\colon\ t \in [a,b])$ it can be given either of the following two forms: 'right' and 'left', denoted respectively by (R) and (L):

For each $\varepsilon > 0$ there is a projection e in $\mathcal{A}$ with $\tau(e) > 1-\varepsilon$, having the property: for each $\eta > 0$ there is $\delta > 0$ such that for any $s,t \in [a,b]$ with $|t-s| < \delta$, we have

$$[X(t) - X(s)]e \in \mathcal{A} \quad \text{and} \quad \|[X(t) - X(s)]e\|_\infty \le \eta, \tag{R}$$

or

$$e[X(t) - X(s)] \in \mathcal{A} \quad \text{and} \quad \|e[X(t) - X(s)]\|_\infty \le \eta, \tag{L}$$

where $\|\cdot\|_\infty$ denotes the norm in the algebra $\mathcal{A}$. This form of 'quantum continuity of paths with probability one', in its right version, has already been considered before in [6, 7], where it was given the name of 'Segal's uniform continuity', and some theorems on this continuity were obtained. However, it is easily seen that the 'right Segal's uniform continuity' which appears in the conclusions of those theorems can be changed to the 'left Segal's uniform continuity', so the results in [6, 7] give in fact both forms of this continuity.

We shall call a process *uniformly continuous in Segal's sense* if it satisfies both (R) and (L) conditions. It is obvious that for a selfadjoint process conditions (R) and (L) are equivalent. Let now ε, e and δ be as above. For arbitrary $s,t \in [a,b]$, $s < t$ choose points $s = t_0 < t_1 < \cdots < t_m = t$ such that $\max_{1\le k\le m}(t_k - t_{k-1}) < \delta$. Then

$$[X(t) - X(s)]e = [X(t) - X(t_{m-1})]e + \cdots + [X(t_1) - X(s)]e,$$

and since all the summands on the right hand side belong to $\mathcal{A}$ we get that $[X(t) - X(s)]e \in \mathcal{A}$. In particular, if $X(s_0) \in \mathcal{A}$ for some $s_0 \in [a, b]$ then right Segal's uniform continuity means that for each $\varepsilon > 0$ there is a projection $e \in \mathcal{A}$ with $\tau(e) > 1 - \varepsilon$ such that the process $(X(t)e\colon t \in [a, b]) \subset \mathcal{A}$ is uniformly continuous in $\|\cdot\|_\infty$-norm. The same holds of course for left Segal's uniform continuity.

Let us say a few words about the terminology. The term 'Segal's convergence' was introduced by E.C. Lance in [9] in honor of I. Segal who first considered this mode of convergence in his celebrated paper [12]. This notion consists in the following: $x_n \to x$ *in Segal's sense* if for each $\varepsilon > 0$ there is a projection $e \in \mathcal{A}$ with $\tau(e^\perp) < \varepsilon$ such that $(x_n - x)e \in \mathcal{A}$ for sufficiently large n, and $\|(x_n - x)e\|_\infty \to 0$. In the definition above it is assumed that τ is a faithful normal *semifinite* trace on $\mathcal{A}$. If τ is finite (as in our case) then Segal's convergence becomes the so-called *almost uniform convergence* (which in the commutative case is *via* Egorov's theorem equivalent to convergence almost everywhere). Now the similarity between Segal's (or in other words: almost uniform) convergence and Segal's continuity is obvious and goes (essentially) like this: in Segal's convergence we can find an 'arbitrarily large' projection e such that $x_n e \to xe$ in $\|\cdot\|_\infty$-norm, while in Segal's continuity we can find an 'arbitrarily large' projection e such that the process $(X(t)e\colon t \in [a, b])$ is uniformly continuous in $\|\cdot\|_\infty$-norm. Accordingly, Segal's continuity might also be called *almost uniform continuity*.

Consider now a process $(X(t)\colon t \in [0, +\infty))$. It is easily seen that the paths of this process are continuous with probability one if and only if for each bounded interval $[a, b]$ contained in $[0, +\infty)$ the paths of the process $(X(t)\colon t \in [a, b])$ are uniformly continuous with probability one. In accordance with the above observation we adopt the following definition.

Definition 1 Let $(X(t)\colon t \in [0, +\infty))$ be a quantum stochastic process. We say that it is (left, right) *continuous in Segal's sense* if for any subinterval $[a, b]$ of the interval $[0, +\infty)$ the process $(X(t)\colon t \in [a, b])$ is (left, right) uniformly continuous in Segal's sense.

The considerations above lead to one more notion of continuity. Namely, the projection e occurring in the definitions of left and right Segal's continuity can be put on both sides. Accordingly, we have

Definition 2 Let $(X(t)\colon t \in [0, +\infty))$ be a quantum stochastic process. We say that it is *weakly continuous in Segal's sense* if for any subinteval $[a, b]$ of the interval $[0, +\infty)$ the process $(X(t)\colon t \in [a, b])$ is *weakly uniformly continuous in Segal's sense*, i.e. for each $\varepsilon > 0$ there is a projection e in $\mathcal{A}$ with $\tau(e) > 1 - \varepsilon$, having the property: for each $\eta > 0$ there is $\delta > 0$ such that for any $s, t \in [a, b]$ with $|t - s| < \delta$, we have

$$e[X(t) - X(s)]e \in \mathcal{A} \quad \text{and} \quad \|e[X(t) - X(s)]e\|_\infty \le \eta.$$

It is clear that both left and right Segal's uniform continuity imply weak Segal's uniform continuity. Moreover, if $(X(t))$ is left and $(Y(t))$ is right uniformly continuous in Segal's sense then $(X(t) + Y(t))$ is weakly uniformly continuous in Segal's sense, while for $(X(t))$ and $(Y(t))$ right (left) uniformly continuous in Segal's sense, the sum $(X(t) + Y(t))$ is right (left) uniformly continuous in Segal's sense (to see it one takes the infimum of the two projections occurring in the definition of Segal's uniform continuity). Obviously, in the commutative case all three modes of continuity are equivalent.

4 Main Result

We begin with a result which may be looked upon as a noncommutative generalization of one of the classical martingale inequalities. Our attention will be restricted to its simplest version for a finite martingale, which suffices for the purposes of this paper; however, it is worth mentioning that a result of this type can be obtained also in a more general setting. The inequality of this type has been obtained by C.J.K. Batty in a slightly different context as 'quantum Kolmogorov's inequality' for sums of 'independent quantum random variables' (cf. [3, Proposition 5.1]).

Proposition *Let* $(X_1, \ldots, X_m)$ *be an* L^2*-martingale. Then for each* $\varepsilon > 0$ *there is a projection* $e \in \mathcal{A}$ *such that*

$$\tau(e^{\perp}) < \frac{\|X_m\|_2^2}{\varepsilon^2},$$

and

$$\|X_n e\|_\infty \leq \varepsilon \quad \textit{for each } n = 1, \ldots, m.$$

The same conclusion holds also in the 'left version' with the projection e put to the left of X_n.

For the proof we refer the reader to [10, Proposition 2].

Now we shall prove our main result on continuity of quantum stochastic processes.

Main Theorem *Let* $(X_n(t)\colon t \in [0, +\infty))$ *be a sequence of* L^2*-martingales* (*left*, *right*, *weakly*) *continuous in Segal's sense*, *such that for each* $t \in [0, +\infty)$

$$\lim_{n\to\infty} X_n(t) = X(t) \quad \textit{in } \|\cdot\|_2\textit{-norm.}$$

Then $(X(t)\colon t \in [0, +\infty))$ *is an* L^2*-martingale* (*left*, *right*, *weakly*) *continuous in Segal's sense.*

Proof That $(X(t))$ is an L^2-martingale follows immediately from the continuity of the conditional expectation in $\|\cdot\|_2$-norm. In proving Segal's continuity we restrict attention to the 'right' case. To this end, we shall prove the right uniform Segal's continuity of $(X(t))$ in an arbitrary interval $[0, a]$.

Let m, n be arbitrary fixed positive integers. The process $(X_n(t) - X_m(t)\colon t \in [0, a])$ is a uniformly right continuous in Segal's sense martingale. For any given $\varepsilon_{nm} > 0$ let f_{nm} be a projection in $\mathcal{A}$ such that $\tau(f_{nm}^{\perp}) < \varepsilon_{nm}$, and the processes $([X_n(t) - X_m(t)]f_{nm}\colon t \in [0, a])$, $(X_n(t)f_{nm}\colon t \in [0, a])$, $(X_m(t)f_{nm}\colon t \in [0, a]) \subset \mathcal{A}$ are uniformly continuous in $\|\cdot\|_\infty$-norm. In particular, there is $\delta_{nm} > 0$ such that for all $t', t'' \in [0, a]$ with $|t' - t''| < \delta_{nm}$ we have

$$\|([X_n(t') - X_m(t')] - [X_n(t'') - X_m(t'')])f_{nm}\|_\infty \leq \frac{\varepsilon_{nm}}{2}. \tag{3}$$

Choose points $0 = t_0 < t_1 < \cdots < t_r = a$ such that $\max_{1\leq i\leq r}(t_i - t_{i-1}) < \delta_{nm}$. For the martingale $(X_n(t_i) - X_m(t_i)\colon i = 0, 1, \ldots, r)$ we infer on account of Proposition that there exists a projection $q_{nm} \in \mathcal{A}$ with

$$\tau(q_{nm}^{\perp}) < \frac{4\|X_n(a) - X_m(a)\|_2^2}{\varepsilon_{nm}^2}$$

such that

$$\|[X_n(t_i) - X_m(t_i)]q_{nm}\|_\infty \le \frac{\varepsilon_{nm}}{2}, \quad \text{for } i = 0, 1, \dots, r. \tag{4}$$

Put $e_{nm} = f_{nm} \wedge q_{nm}$. Then

$$\tau(e_{nm}) < \varepsilon_{nm} + \frac{4\|X_n(a) - X_m(a)\|_2^2}{\varepsilon_{nm}^2},$$

and for each $t \in [0, a]$ there is t_i such that $|t - t_i| < \delta_{nm}$, so from (3) and (4) we get

$$\begin{aligned}
&\|[X_n(t) - X_m(t)]e_{nm}\|_\infty \\
&\quad \le \|([X_n(t) - X_m(t)] - [X_n(t_i) - X_m(t_i)])e_{nm}\|_\infty + \|[X_n(t_i) - X_m(t_i)]e_{nm}\|_\infty \\
&\quad \le \|([X_n(t) - X_m(t)] - [X_n(t_i) - X_m(t_i)])f_{nm}\|_\infty + \|[X_n(t_i) - X_m(t_i)]q_{nm}\|_\infty \\
&\quad \le \frac{\varepsilon_{nm}}{2} + \frac{\varepsilon_{nm}}{2} = \varepsilon_{nm}.
\end{aligned}$$

Moreover, the processes $(X_n(t)e_{nm}\colon t \in [0, a])$ and $(X_m(t)e_{nm}\colon t \in [0, a])$ are uniformly continuous in $\|\cdot\|_\infty$-norm.

Let an arbitrary $\varepsilon > 0$ be given. We have $X_n(a) \to X(a)$ in $\|\cdot\|_2$-norm, so we can find a subsequence $\{n_k\}$ such that

$$\|X_{n_{k+1}}(a) - X_{n_k}(a)\|_2 < \frac{\varepsilon^3}{2^{3k+5}}, \quad k = 1, 2, \dots.$$

Apply our previous considerations to the martingale $(X_{n_{k+1}}(t) - X_{n_k}(t)\colon t \in [0, a])$ (i.e. we take $n = n_{k+1}$, $m = n_k$). For $(\varepsilon_{nm} =)\ \varepsilon_k = \varepsilon/2^{k+1}$ there is a projection $e_k \in \mathcal{A}$ with

$$\tau(e_k^\perp) < \frac{\varepsilon}{2^{k+1}} + \frac{4(2^{k+1})^2\|X_{n_{k+1}}(a) - X_{n_k}(a)\|_2^2}{\varepsilon^2} < \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2^k},$$

such that for each $t \in [0, a]$

$$\|[X_{n_{k+1}}(t) - X_{n_k}(t)]e_k\|_\infty \le \varepsilon_k = \frac{\varepsilon}{2^{k+1}}.$$

Put

$$e = \bigwedge_{k=1}^{\infty} e_k.$$

Then $\tau(e^\perp) < \varepsilon$, and for each $t \in [0, a]$

$$\|[X_{n_{k+1}}(t) - X_{n_k}(t)]e\|_\infty \le \|[X_{n_{k+1}}(t) - X_{n_k}(t)]e_k\| \le \varepsilon_k = \frac{\varepsilon}{2^{k+1}}. \tag{5}$$

Since the processes $(X_{n_k}(t)e_k\colon t \in [0, a])$, $k = 1, 2, \dots$ are uniformly continuous in $\|\cdot\|_\infty$-norm it follows that the processes $(X_{n_k}(t)e\colon t \in [0, a])$, $k = 1, 2, \dots$ are also uniformly continuous in $\|\cdot\|_\infty$-norm. Condition (5) says that the sequence of processes $(X_{n_k}(t)e\colon t \in [0, a])$, $k = 1, 2, \dots$ is Cauchy in $\|\cdot\|_\infty$-norm uniformly for $t \in [0, a]$. Since

$$X_{n_k}(t)e \to X(t)e \quad \text{in } \|\cdot\|_2\text{-norm},$$

it follows that

$$X_{n_k}(t)e \to X(t)e \quad \text{in } \|\cdot\|_\infty\text{-norm}$$

uniformly for $t \in [0,a]$, and the $\|\cdot\|_\infty$-norm continuity of $(X_{n_k}(t)e\colon t \in [0,a])$ yields the norm continuity of $(X(t)e\colon t \in [0,a])$ which proves the claim. □

5 Applications

The first group of applications of the Main Theorem concerns stochastic integrals. For a more detailed presentation of the theories of integrals to which our result can be applied the reader is referred to [1, 2, 5, 10, 11]. Here we only indicate the main points. Let $(X(t)\colon t \in [0,+\infty))$ be an L^2-martingale, and let $f : [0,+\infty) \to \mathcal{A}$ be an *adapted* map, by which is meant that for each $t \geq 0$, $f(t) \in \mathcal{A}_t$. The stochastic integrals

$$\int_0^t f(u)dX(u) \quad \text{and} \quad \int_0^t dX(u)f(u)$$

are defined as L^2-limits of the sums

$$S_n^r(t) = \sum_{k=1}^{k_n} f(t_{k-1}^{(n)})[X(t_k^{(n)}) - X(t_{k-1}^{(n)})]$$

and

$$S_n^l(t) = \sum_{k=1}^{k_n} [X(t_k^{(n)}) - X(t_{k-1}^{(n)})]f(t_{k-1}^{(n)}),$$

respectively, where $0 = t_0^{(n)} < t_1^{(n)} \cdots < t_{k_n}^{(n)} = t$ is a sequence of partitions of the interval $[0,t]$. From the Main Theorem we obtain the following counterpart of the classical result on continuity of stochastic integrals (cf. [4, Theorem 2.6] and [8, Chap. III, Sect. 2]).

Theorem 1 *Let* $(X(t)\colon t \in [0,+\infty))$ *be an* L^2*-quantum martingale, and let* $f : [0,+\infty) \to \mathcal{A}$ *be an adapted map such that for each* $t \in [0,+\infty)$ *the stochastic integrals*

$$Y(t) = \int_0^t f(u)dX(u) \quad \textit{and} \quad Z(t) = \int_0^t dX(u)f(u)$$

exist. Then if $(X(t))$ *is right continuous in Segal's sense,* $(Y(t))$ *is right continuous in Segal's sense, while for* $(X(t))$ *left continuous in Segal's sense,* $(Z(t))$ *is left continuous in Segal's sense.*

For the proof it suffices to notice that for $(X(t))$ right (respectively left) continuous in Segal's sense the processes $(S_n^r(t)\colon t \in [0,+\infty))$ (respectively $(S_n^l(t)\colon t \in [0,+\infty))$) are right (respectively left) continuous in Segal's sense L^2-martingales, and to apply the Main Theorem.

The next example of use of our result consists, roughly speaking, in showing that the space of martingales (left, right, weakly) continuous in Segal's sense is closed. More precisely, let $\mathcal{M}_2$ denote the space of all L^2-martingales, and let $\mathcal{M}_2^c$ stand for continuous in

Segal's sense L^2-martingales. In $\mathcal{M}_2$ we define a metric ρ by the formula

$$\rho(X,Y)=\sum_{n=1}^{\infty}\frac{\|X(n)-Y(n)\|_2\wedge 1}{2^n},$$

where $X=(X(t)\colon t\in[0,+\infty))$, $Y=(Y(t)\colon t\in[0,+\infty))$ (cf. [8, Chap. I, Sect. 5]). Then we have the following counterpart of Proposition 5.23 in [8].

Theorem 2 *Under the metric ρ, $\mathcal{M}_2$ is a complete metric space, and $\mathcal{M}_2^c$ is a closed subspace of $\mathcal{M}_2$.*

Proof Let $(M(t)\colon t\in[0,+\infty))$ be an arbitrary L^2-martingale. For $s\le t$ we have on account of (1) and (2)

$$\tau(|M(t)|^2)=\tau(\mathbb{E}_s|M(t)|^2)\ge\tau(|\mathbb{E}_sM(t)|^2)=\tau(|M(s)|^2)$$

and thus the function $[0,+\infty)\ni t\mapsto\|M(t)\|_2$ is nondecreasing. It follows that the convergence $X_n\to X$ with respect to the metric ρ is equivalent to pointwise convergence $X_n(t)\to X(t)$, $t\in[0,+\infty)$, in the norm $\|\cdot\|_2$. Now the first part of the theorem follows from the continuity of the conditional expectations $\mathbb{E}_t$ in $L^2(\mathcal{A})$, while the second is a consequence of the Main Theorem. □

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